Rational Root Theorem and Solving Polynomials

Name: Key

Rational Root Theorem: For every polynomial of the form $P(x) = ax^n + bx^{n-1} + \cdots + c$, where a, b, c are integers and $a, c \neq 0$, if $p(x)$ has a rational root of the reduced form $\frac{p}{q}$, then p is a factor of the constant term c and q is a factor of the leading coefficient a.

Possible Rational Roots $=$ $\pm \frac{\text{Factors of the constant term}}{\text{Factor of the leading coefficient}}$

List all the possible rational roots for each of the following polynomials.

1. $y = 5x^4 - 8x^2 - 4$

1, 5 | 1, 2, 4

$\pm 1, \pm 2, \pm 4, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{4}{5}$

2. $y = 12x^3 + 4x^2 - 6x - 2$

1, 2, 3, 4, 6, 12 | 1, 2

$\pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \pm\frac{1}{6}, \pm\frac{1}{12}, \pm 2, \pm\frac{2}{3}$

3. $y = 10x^4 - 40$

1, 2, 5, 10 | 1, 2, 4, 5, 8, 10, 20, 40

$\pm 1, \pm\frac{1}{2}, \pm\frac{1}{5}, \pm\frac{1}{10}, \pm 2, \pm 4, \pm\frac{2}{5}, \pm\frac{4}{5}, \pm 5, \pm\frac{5}{2}, \pm 8, \pm\frac{8}{5}, \pm 10, \pm 40, \pm 20$

First discuss the nature of the zeros and then find all the zeros of the functions below.

4. $y = 5x^4 - 8x^2 - 4$ — 4 total: 4, 2 or 0 real; 4, 2, 0 imag.

$0 = (5x^2+2)(x^2-2)$

$5x^2+2=0 \quad x^2-2=0$

$x^2 = -\frac{2}{5} \quad x^2 = 2$

$x = \pm i\sqrt{\frac{2}{5}} \quad x = \pm\sqrt{2}$

5. $y = 12x^3 + 4x^2 - 6x - 2$ — 3 total: 3, 1 real; 0, 2 imag.

$0 = 4x^2(3x+1) - 2(3x+1)$

$0 = (4x^2-2)(3x+1)$

$4x^2-2=0 \quad 3x+1=0$

$x = \pm\sqrt{\frac{1}{2}} \quad x = -\frac{1}{3}$

6. $y = 10x^4 - 40$ — 4 total; 4 or 2 real; 0 or 2 imag.

$= 10(x^4-4)$

$= 10(x^2-2)(x^2+2)$

$x = \pm\sqrt{2} \quad x = \pm i\sqrt{2}$

Solve each of the following. Find all real and imaginary solutions.

7. $-64x^3 + 125 = 0$

$(4x+5)(16x^2+20x+25)$

$x = \frac{5}{4}$

$x = \frac{-20 \pm \sqrt{400-4(16)(25)}}{32}$

$x = \frac{-20 \pm 20i\sqrt{3}}{32} = \frac{-5 \pm 5i\sqrt{3}}{8}$

8. $2x^3 - 13x^2 + 20x = 0$

$x(2x^2-13x+20)=0$

$x(2x-5)(x-4)$

$x=0 \quad x=\frac{5}{2} \quad x=4$

9. $x^3 + x = 0$

$x(x^2+1)=0$

$x=0 \quad x=\pm i$

10. $4x^3 - 7x^2 = 3x^2$

$4x^3 - 10x^2 = 0$

$2x^2(2x-5)=0$

$x=0 \quad x=\frac{5}{2}$

mult. 2

11. $x^4 - 64 = 0$

$(x^2-8)(x^2+8)$

$x = \pm 2\sqrt{2} \quad x = \pm 2i\sqrt{2}$

12. $5x^5 - 80x = 0$

$5x(x^4-16)=0$

$5x(x^2-4)(x^2+4)=0$

$5x(x+2)(x-2)(x^2+4)=0$

$x=0 \quad x=-2 \quad x=2 \quad x=\pm 2i$

13. $8x^5 + 5x^2 = 4x^2$

$8x^5 + x^2 = 0$

$x^2(8x^3+1)=0$

$x^2(2x+1)(4x^2-2x+1)=0$

$x=0 \quad x=-\frac{1}{2} \quad x = \frac{2 \pm \sqrt{4-4(4)(1)}}{8} = \frac{2 \pm \sqrt{-12}}{8} = \frac{2 \pm 2i\sqrt{3}}{8} = \frac{1 \pm i\sqrt{3}}{4}$

mult. 2

14. $16x^3 = 54$

$16x^3 - 54 = 0$

$2(8x^3-27)=0$

$2(2x-3)(4x^2+6x+9)=0$

$x = \frac{3}{2}$

$x = \frac{-6 \pm \sqrt{36-4(4)(9)}}{8}$

$x = \frac{-6 \pm 6i\sqrt{3}}{8} = \frac{-3 \pm 3i\sqrt{3}}{4}$

15. $x^4 + x^2 - 42 = 0$

$(x^2+7)(x^2-6)=0$

$x = \pm i\sqrt{7} \quad x = \pm\sqrt{6}$

List all the possible rational roots for each of the following polynomials.

1. $y = 12x^3 - 3x^2$

$x^2(12x-3)$ ~~$3x^2(4x-1)$~~

1, 3 | 1, 2, 3, 4, 6, 12

$\pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \pm\frac{1}{6}, \pm\frac{1}{12}$

$\pm 3, \pm\frac{3}{2}, \pm\frac{3}{4}$

2. $y = 9x^4 - 16$

1, 3, 9 | 1, 2, 4, 8, 16

$\pm 1, \pm\frac{1}{3}, \pm\frac{1}{9}$

$\pm 2, \pm\frac{2}{3}, \pm\frac{2}{9}$

$\pm 4, \pm\frac{4}{3}, \pm\frac{4}{9}$

$\pm 8, \pm\frac{8}{3}, \pm\frac{8}{9}$

$\pm 16, \pm\frac{16}{3}, \pm\frac{16}{9}$

3. $y = 2x^3 - 16x^2 - 4x + 32$

~~$2x^2(x-8) - 4(x-8)$~~

1, 2 | 1, 2, 4, 8, 16, 32

$\pm 1, \pm\frac{1}{2}, \pm 2$

$\pm 4, \pm 8, \pm 16$

± 3 2

First discuss the nature of the zeros, then find each of the zeros for the functions below.

4. $f(x) = 12x^3 - 3x^2$

3 total real: 3, 1 imag: 0 or 2

$3x^2(4x-1)$

$x=0$ $x=\frac{1}{4}$

5. $f(x) = 9x^4 - 16$

total: 4 real: 4, 2, 0 imag: 4, 2, 0

$0 = (3x^2-4)(3x^2+4)$

$x = \pm\sqrt{4/3}$ $x = \pm i\sqrt{4/3}$

6. $f(x) = 2x^3 - 16x^2 - 4x + 32$

total: 3 real: 3, 1 imag: 0, 2

$2x^2(x-8) - 4(x-8)$

$2(x^2-2)(x-8)$

$x = \pm\sqrt{2}$ $x = 8$

Solve each of the following. Find all real and imaginary solutions.

7. $2x^3 - 5x^2 + 40x - 100 = 0$

$x^2(2x-5) + 20(2x-5) = 0$

$(x^2+20)(2x-5) = 0$

$x = \pm 2i\sqrt{5}$ $x = 5/2$

8. $x^3 + 3x^2 = 24x + 72$

$x^3 + 3x^2 - 24x - 72 = 0$

$x^2(x+3) - 24(x+3) = 0$

$(x^2-24)(x+3) = 0$

$x = \pm 2\sqrt{6}$ $x = -3$

9. $4x^4 + 35x^2 - 9 = 0$

$(4x^2-1)(x^2+9) = 0$

$x = \frac{1}{2}$ $x = -9$

10. $x^4 - 16x^2 = x^2 + 18$

$x^4 - 17x^2 - 18 = 0$

$(x^2-18)(x^2+1)$

$x = \pm\sqrt{18}$ $x = \pm i$

$= \pm 3\sqrt{2}$

11. $2x^3 - 16x^2 - 40x = 0$

$2x(x^2 - 8x - 20) = 0$

$2x(x-10)(x+2) = 0$

$x = 0$ $x = 10$ $x = -2$

12. $81x^4 = 3x$

$81x^4 - 3x = 0$

$3x(27x^3 - 1) = 0$

$3x(3x-1)(9x^2+3x+1)$

$x = 0$ $x = \frac{1}{3}$ $x = \frac{-3 \pm \sqrt{9-4(9)(1)}}{18}$

$= \frac{-3 \pm \sqrt{-27}}{18} = \frac{-3 \pm 3i\sqrt{3}}{18} = \frac{-1 \pm i\sqrt{3}}{6}$

13. $8x^3 - 125 = 0$

$(2x-5)(4x^2+10x+25) = 0$

$x = 5/2$ $x = \frac{-10 \pm \sqrt{100-4(4)(25)}}{2(4)}$

$x = \frac{-10 \pm \sqrt{-300}}{8} = \frac{-10 \pm 10i\sqrt{3}}{8} = \frac{-5 \pm 5i\sqrt{3}}{4}$

14. $x^3 + 512 = 0$

$(x+8)(x^2 - 8x + 64)$

$x = -8$ $x = \frac{8 \pm \sqrt{64-4(1)(64)}}{2}$

$x = \frac{8 \pm 8\sqrt{3}}{2} = 4 \pm 4\sqrt{3}$

15. $2x^4 = 9x^2$

$2x^4 - 9x^2 = 0$

$x^2(2x^2 - 9) = 0$

$x = 0$ mult 2

$x = \pm\sqrt{9/2} = \pm\frac{3}{\sqrt{2}}$

16. The population of a species is modeled by the equation $p(t) = -t^4 + 72t^2 + 225$, where t is the number of years. Find the approximate number of years until the species is extinct.

$0 = -t^4 + 72t^2 + 225$

$0 = (-t^2 + 75)(t^2 + 3)$

$t = \pm 5\sqrt{3}$ $t = \pm i\sqrt{3}$

$t = \sqrt{75}$

≈ 8.66 years